Finite Element Introduction (Slides) 2011/9/16 Jing Du. ①

· Applications.

Elasticity: continuity, homogeneity, isotropy, linear elasticity, small deformation

3 Variables, 3 equations, Boundary Conditions.

displacement, u_i

strain, ε_{ij}

stress, σ_{ij}

equilibrium : $\sigma_{ij,j} + \bar{b}_i = 0.$ (body force)

compatibility : $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

constitutive : $\varepsilon_{ij} = D_{ijkl}\sigma_{kl}$ ~~or~~ $\sigma_{ij} = D_{ijkl}\varepsilon_{kl}$. (stiffness ~~compliance~~)

BC: $\begin{cases} u_i = \bar{u}_i \text{ on } S_u \\ \sigma_{ij} n_j = \bar{p}_i \text{ on } S_p. \end{cases}$

$$U_i = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad \varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \frac{1}{2}\gamma_{xy} \\ \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} \end{bmatrix} \quad \sigma_{ij} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \cdots \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

$$\begin{cases} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \bar{b}_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \bar{b}_y = 0 \\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \bar{b}_z = 0, \end{cases}$$

$$\begin{cases} \varepsilon_{xx} = \frac{\partial u}{\partial x}, \\ \varepsilon_{yy} = \frac{\partial v}{\partial y} \\ \varepsilon_{zz} = \frac{\partial w}{\partial z} \\ \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\ \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}. \end{cases}$$

$$\begin{cases} \varepsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \mu(\sigma_{yy} + \sigma_{zz})] \\ \varepsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \mu(\sigma_{xx} + \sigma_{zz})] \\ \varepsilon_{zz} = \frac{1}{E}[\sigma_{zz} - \mu(\sigma_{xx} + \sigma_{yy})] \\ \gamma_{xy} = \frac{1}{G}\tau_{xy} \\ \gamma_{yz} = \frac{1}{G}\tau_{yz} \\ \gamma_{zx} = \frac{1}{G}\tau_{zx} \end{cases}$$

$\begin{cases} u = \bar{u} \\ v = \bar{v} \\ w = \bar{w} \end{cases}$ on S_u

$\begin{cases} \sigma_{xx} n_x + \tau_{xy} n_y + \tau_{zx} n_z = \bar{p}_x \\ \tau_{xy} n_x + \sigma_{yy} n_y + \tau_{yz} n_z = \bar{p}_y \\ \tau_{zx} n_x + \tau_{yz} n_y + \sigma_{zz} n_z = \bar{p}_z \end{cases}$ on S_p.

plane stress, plane strain simplification.

- ~~Variab~~ Minimum potential energy / Variational Principle / Rayleigh-Ritz method
- Weighted-Residual / Galerkin approximation

beam: Solve for $v(x)$, $L(v(x)) + \bar{b} = 0$. Operator $L(\ ) = -EI\frac{d^4}{dx^4}$, $\bar{b} = \bar{p}(x)$

BC(u): $g_u(v(x)) = 0$ on S_u

BC(p): $g_p(v(x)) = 0$ on S_p.

Assume $\hat{v}(x)$ satisfies BC(u) and BC(p), $\hat{v}(x) = C_1\phi_1(x) + C_2\phi_2(x) + \cdots + C_n\phi_n(x)$

unknowns bases

residual error $R = L(\hat{v}(x)) + \bar{b} \neq 0$.

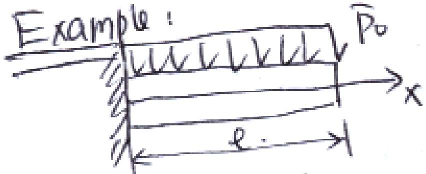
$$\begin{cases} \int_\Omega w_{t1} \cdot R(C_1, C_2, \ldots C_n, \phi_1, \phi_2, \ldots \phi_n)\, d\Omega = 0 \\ \int_\Omega w_{t2} \cdot R(C_1, C_2, \ldots, C_n, \phi_1, \phi_2, \ldots \phi_n)\, d\Omega = 0 \\ \vdots \\ \int_\Omega w_{tn} \cdot R(C_1, C_2, \ldots C_n, \phi_1, \phi_2, \ldots, \phi_n)\, d\Omega = 0 \end{cases}$$

weight function.

Solve for $C_1, C_2, \ldots, C_n \Rightarrow \hat{v}(x)$.

Take $w_{t1}, w_{t2}, \ldots w_{tn}$ as $\phi_1, \phi_2 \ldots \phi_n \Rightarrow$ Galerkin method.

Example:



cantilever beam

uniform pressure.

$$BC(u): \begin{cases} v|_{x=0} = 0 \\ v'|_{x=0} = 0 \end{cases}$$

$$BC(p): \begin{cases} M = -EI v''|_{x=l} = 0 \\ Q = -EI v'''|_{x=l} = 0 \end{cases}$$

- minimum potential energy

$\hat{v}(x) = c(1 - \cos\frac{\pi x}{2l})$, satisfies BC(u), solve for c.

potential energy. $\Pi = \int_0^l \frac{1}{2} EI(\hat{v}'')^2 dx - \int_0^l \bar{p}_0 \hat{v}\, dx - \bar{p}_0 c \int_0^l (1 - \cos\frac{\pi x}{2l}) dx$.

$$= \frac{1}{2} EI \left(\frac{\pi}{2l}\right)^4 \cdot c^2 \int_0^l \cos^2\frac{\pi x}{2l} dx - \bar{p}_0 c \int_0^l \left(1-\cos\frac{\pi x}{2l}\right) dx$$

$$= \frac{1}{2} C^2 EI \left(\frac{\pi}{2l}\right)^4 \left(\frac{l}{2}\right) - \bar{p}_0 c l \left(1 - \frac{2}{\pi}\right).$$

minimum at $\frac{\partial \Pi}{\partial c} = 0 \Rightarrow c = \frac{32}{\pi^4}\left(1 - \frac{2}{\pi}\right)\frac{\bar{p}_0 l^4}{EI}$.

maximum deflection at $x = l$, $\hat{v}|_{x=l} = c = 0.11937 \frac{\bar{p}_0 l^4}{EI}$.

To be more accurate, $\hat{v} = \sum_{n=1}^{N} C_n \left(1 - \cos\frac{(2n-1)\pi x}{2l}\right)$. ? $N=5$.

- Galerkin Method. $\hat{v}(x) = \sum_{n=1}^{N} c_n \phi_n(x)$.

$\int \phi_n (EI \hat{v}^{(4)} - \bar{p}_0)\, dx = 0, \ (n = 1, 2, \ldots, N)$.

$\hat{v}(x)$ should satisfy both BC(u) and BC(p).

BC(p) $\Rightarrow$ $\frac{d^2\hat{v}}{dx^2} = c\left(1 - \sin\frac{\pi x}{2l}\right)$. satisfies BC(p)

Integral twice. $\hat{v}(x) = c\left[\frac{x^2}{2} + \left(\frac{2l}{\pi}\right)^2 \sin\frac{\pi x}{2l} + Ax + B\right]$.

solve for A & B to satisfy BC(u)., $A = -2\ell/\pi$, $B=0$.

$$\hat{v}(x) = c\left[\frac{x^2}{2} - \frac{2\ell}{\pi}x + \left(\frac{2\ell}{\pi}\right)^2 \sin\frac{\pi x}{2\ell}\right] = c\phi(x).$$

$$\int_0^\ell \left[EI\, c\left(\frac{\pi}{2\ell}\right)^2 \sin\frac{\pi x}{2\ell} - \bar{p}_0\right]\left[\frac{x^2}{2} - \frac{2\ell}{\pi}x + \left(\frac{2\ell}{\pi}\right)^2 \sin\frac{\pi x}{2\ell}\right] dx = 0.$$

$$c = \frac{\frac{1}{6} - \frac{1}{\pi} + \frac{8}{\pi^3}}{\frac{3}{2} - \frac{4}{\pi}} \frac{\bar{p}_0 \ell^2}{EI} = 0.469 \frac{\bar{p}_0 \ell^2}{EI}.$$

$$\hat{v}\big|_{x=\ell} = c\ell\left(\frac{1}{2} - \frac{2}{\pi} + \frac{4}{\pi^2}\right) = 0.126 \frac{\bar{p}_0 \ell^2}{EI}.$$

same ? for minimum potential energy. (same answer)

Finite Element: discretization $\Omega = \sum \Omega^e$.

~~a element of n nodes. DOF~~

nodal displacement vector ~~of element~~ q^e

~~displacement~~ force ~~of element~~ p^e

Guess u. from lower order to higher order, unique

$u(x,y,z) = a_0 + a_1 x + a_2 y + a_3 z + a_4 xy + a_5 yz + a_6 zx + \cdots$

$u(\xi) = a_0 + a_1 \xi + \cdots + a_{n-1}\xi^{n-1}$

~~shape function matrix of element N~~

$u^e = N \cdot q^e$; where N is shape function matrix of element

$\varepsilon = [\partial]u = [\partial]N q^e = B q^e$; where B is strain-displacement matrix

stiffness matrix of element $K^e = \int_{\Omega^e} B^T D B \, d\Omega$.

force matrix $P^e = \int_{\Omega^e} N^T \bar{b}\, d\Omega + \int_{S_p^e} N^T \bar{p}\, dA$.

stiffness equation of element $K^e q^e = P^e$.

define coordinate transformation matrix of element T^e

$\bar{K}^e = T^{eT} \cdot K^e \cdot T^e$. $\Rightarrow$ $\bar{K}^e \cdot \bar{q}^e = \bar{P}^e$

$\Rightarrow$ $\bar{P}^e = T^{eT} P^e$. stiffness equation of element in global coordinate.

$q^e = T^e \cdot \bar{q}^e$

global ~~stiffness~~ equation $K \cdot q = P$, where $K = \sum \bar{K}^e$, $q = \sum q^e$, $P = \sum P^e$.
Assembly

Apply BC. solve for unknowns (go to next page.)

$\sigma^e = D^e B^e q^e$, $\varepsilon^e = B q^e$, $\Pi^e = \frac{1}{2} q^{eT} K^e q^e - P^{eT} q^e$.

$q = [q_u \;\; \bar{q}_k]^T, \quad \mathbb{P} = [\bar{F}_k \;\; \mathbb{R}_u]^T.$

$$\begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}\begin{bmatrix} q_u \\ \bar{q}_k \end{bmatrix} = \begin{bmatrix} \bar{F}_k \\ R_u \end{bmatrix}$$

$$q_u = K_1^{-1}[\bar{F}_k - K_2 \bar{q}_k]$$

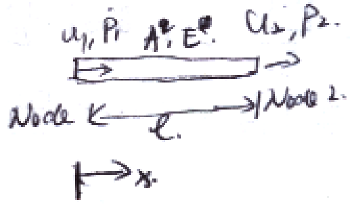
$$\mathbb{R}_u = K_3 q_u + K_4 \bar{q}_k$$

<u>Truss</u>

DOF = 2.

$q^e = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\mathbb{P}^e = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$



$$u(x) = a_0 + a_1 x$$

$$\left.\begin{aligned} u(0) &= a_0 = u_1 \\ u(\ell) &= a_0 + a_1 \ell = u_2 \end{aligned}\right\} \Rightarrow \begin{aligned} a_0 &= u_1 \\ a_1 &= \frac{u_2 - u_1}{\ell} \end{aligned}$$

$$u(x) = u_1 + \frac{u_2 - u_1}{\ell} x = \left(1 - \frac{x}{\ell}\right) u_1 + \left(\frac{x}{\ell}\right) u_2 = \mathbb{N} \cdot q^e$$

$$\mathbb{N} = \left[1 - \frac{x}{\ell} \quad \frac{x}{\ell}\right]$$

(1×2).

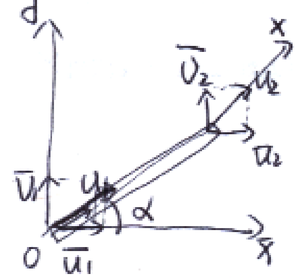
$$\varepsilon^e(x) = \frac{du(x)}{dx} = \left[-\frac{1}{\ell} \quad \frac{1}{\ell}\right]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbb{B}(x) q^e$$

where $\mathbb{B}(x) = \frac{d}{dx}\mathbb{N}(x) = \left[-\frac{1}{\ell} \quad \frac{1}{\ell}\right]$

$$\sigma^e(x) = E\, \mathbb{B}(x) q^e.$$

$$\mathbb{K}^e = \int_0^\ell A \, \mathbb{B}^T \mathbb{D} \mathbb{B} A\, dx = \int_0^\ell A \begin{bmatrix} -\frac{1}{\ell} \\ \frac{1}{\ell} \end{bmatrix} E \left[-\frac{1}{\ell} \quad \frac{1}{\ell}\right] dx = \frac{EA}{\ell}\begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} u_1 \\ u_2 \end{array}$$

(Truss on a Plane. ~~Problem~~)



$q^e = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. $\quad \bar{q}^e = \begin{bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{bmatrix}$

$$\left.\begin{aligned} u_1 &= \bar{u}_1 \cos\alpha + \bar{v}_1 \sin\alpha. \\ u_2 &= \bar{u}_2 \cos\alpha + \bar{v}_2 \sin\alpha \end{aligned}\right\}.$$

$$q^e = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix}\begin{bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{bmatrix} = \mathbb{T}^e \cdot \bar{q}^e$$

Then, $\mathbb{T}^e = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix}$

$$\bar{\mathbb{K}}^e = \frac{EA}{\ell}\begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha & -\cos^2\alpha & -\cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha & -\cos\alpha\sin\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -\cos\alpha\sin\alpha & \cos^2\alpha & \cos\alpha\sin\alpha \\ -\cos\alpha\sin\alpha & -\sin^2\alpha & \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix}$$

$$= \mathbb{T}^{eT} \mathbb{K}^e \mathbb{T}^e$$

$V = V_1 \hat{x} + V_2 \hat{y} + V_3 \hat{z}$

$\cos\alpha = \frac{V \cdot \hat{x}}{\|V\|} = \frac{V_1}{\sqrt{V_1^2 + V_2^2 + V_3^2}}$

direction cosine.

(Truss in 3D ~~Problem~~ space)

$$\mathbb{T}^e = \begin{bmatrix} \cos(x,\bar{x}) & \cos(x,\bar{y}) & \cos(x,\bar{z}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(x,\bar{x}) & \cos(x,\bar{y}) & \cos(x,\bar{z}) \end{bmatrix}$$

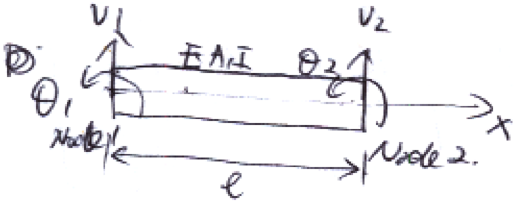
where $\cos(x,\bar{x}) = \frac{\bar{x}_2 - \bar{x}_1}{\ell}$

$\cos(x,\bar{y}) = \frac{\bar{y}_2 - \bar{y}_1}{\ell}$

$\cos(x,\bar{z}) = \frac{\bar{z}_2 - \bar{z}_1}{\ell}$.

Beam

(Pure bending).



DOF $= 4$. $\mathbb{q}^e = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$, $\quad \mathbb{P}^e = \begin{bmatrix} P_{v1} \\ M_1 \\ P_{v2} \\ M_2 \end{bmatrix}$

$$v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Plug in BC $\begin{cases} v(0) = v_1, \ v'(0) = \theta_1 \\ v(\ell) = v_2, \ v'(\ell) = \theta_2. \end{cases} \Rightarrow \begin{cases} a_0 = v_1 \\ a_1 = \theta_1 \\ a_2 = \frac{1}{\ell^2}(-3v_1 - 2\theta_1 \ell + 3v_2 - \theta_2 \ell) \\ a_3 = \frac{1}{\ell^3}(2v_1 + \theta_1 \ell - 2v_2 + \theta_2 \ell) \end{cases}$

$$v(x) = (1 - 3\xi^2 + 2\xi^3) v_1 + \ell(\xi - 2\xi^2 + \xi^3)\theta_1 + (3\xi^2 - 2\xi^3) v_2 + \ell(\xi^3 - \xi^2)\theta_2 = \mathbb{N}(\xi)\mathbb{q}^e$$

([illegible]) where $\xi = \frac{x}{\ell}$. $\mathbb{N}(\xi) = [(1 - 3\xi^2 + 2\xi^3) \quad \ell(\xi - 2\xi^2 + \xi^3) \quad (3\xi^2 - 2\xi^3) \quad \ell(\xi^3 - \xi^2)]$

$$\varepsilon(x, \hat{y}) = -\hat{y}\frac{d^2 v(x)}{dx^2} = -\hat{y}\left[\frac{1}{\ell^2}(12\xi - 6) \quad \frac{1}{\ell}(6\xi - 4) \quad -\frac{1}{\ell^2}(12\xi - 6) \quad \frac{1}{\ell}(6\xi - 2)\right] \cdot \mathbb{q}^e = \mathbb{B}(\xi) \cdot \mathbb{q}^e.$$

where $\hat{y}$ is y coordinate starting from neutral ~~layer~~ plane.

~~$\mathbb{B} = -\hat{y}[B_1 \ B_2 \ B_3 \ B_4]$.~~

$$\mathbb{B} = -\hat{y}\left[\frac{1}{\ell^2}(12\xi - 6); \quad \frac{1}{\ell}(6\xi - 4) \quad -\frac{1}{\ell^2}(12\xi - 6) \quad \frac{1}{\ell}(6\xi - 2)\right] = -\hat{y}[B_1 \ B_2 \ B_3 \ B_4]$$

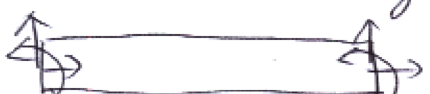
$$\sigma(x, \hat{y}) = E\,\varepsilon(x, \hat{y}) = E\,\mathbb{B}(x, \hat{y}) \cdot \mathbb{q}^e.$$

$$\mathbb{K}^e = \int_0^\ell \int_A \mathbb{B}^T \mathbb{D}\, \mathbb{B}\, dA\, dx = \int_0^\ell \int_A (-\hat{y}) \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} E \cdot [B_1 \ B_2 \ B_3 \ B_4](-\hat{y}) \cdot dA\, dx.$$

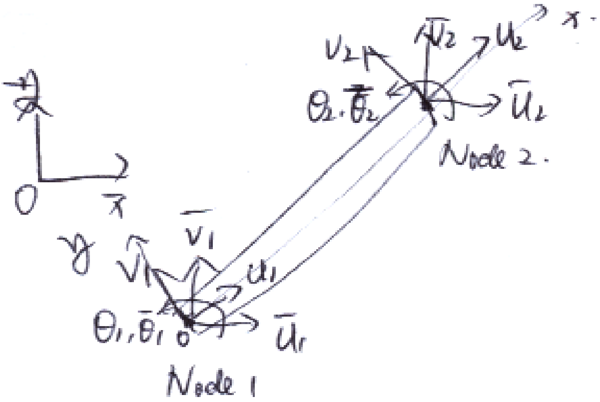
$$= \frac{EI}{\ell^3} \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix} \end{array} \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2. \end{array}$$

(Bending + Tension) (~~small deform~~ linear elasticity).

DOF $= 6$.



$$\mathbb{K}^e = \begin{bmatrix} \frac{EA}{\ell} & 0 & 0 & -\frac{EA}{\ell} & 0 & 0 \\ 0 & \frac{12EI}{\ell^3} & \frac{6EI}{\ell^2} & 0 & -\frac{12EI}{\ell^3} & \frac{6EI}{\ell^2} \\ 0 & \frac{6EI}{\ell^2} & \frac{4EI}{\ell} & 0 & -\frac{6EI}{\ell^2} & \frac{2EI}{\ell} \\ -\frac{EA}{\ell} & 0 & 0 & \frac{EA}{\ell} & \frac{12EI}{\ell^3} & -\frac{6EI}{\ell^2} \\ 0 & -\frac{12EI}{\ell^3} & -\frac{6EI}{\ell^2} & 0 & & \\ 0 & \frac{6EI}{\ell^2} & \frac{2EI}{\ell} & 0 & -\frac{6EI}{\ell^2} & \frac{4EI}{\ell} \end{bmatrix} = \begin{array}{c} \begin{array}{cccccc} u_1 \ 支 & 曲\ v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} \boxed{t} & 0 & & \boxed{I} & 0 & \\ 0 & b & & 0 & b & \\ \boxed{t} & 0 & & \boxed{I} & 0 & \\ 0 & b & & 0 & b & \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2. \end{array}$$



$$q^e = \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{bmatrix} \qquad \bar{q}^e = \begin{bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{u}_1 \\ \bar{v}_1 \\ \bar{\theta}_1 \end{bmatrix}$$

$$\begin{cases} u_1 = \bar{u}_1 \cos\alpha + \bar{v}_1 \sin\alpha \\ v_1 = -u_1 \sin\alpha + \bar{v}_1 \cos\alpha \\ \theta_1 = \bar{\theta}_1 \\ u_2 = \bar{u}_2 \cos\alpha + \bar{v}_2 \sin\alpha \\ v_2 = -\bar{u}_2 \sin\alpha + \bar{v}_2 \cos\alpha \\ \theta_2 = \bar{\theta}_2 \end{cases}$$

$$q^e = T^e \cdot \bar{q}^e \qquad T^e = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & & & \\ -\sin\alpha & \cos\alpha & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ & & & \cos\alpha & \sin\alpha & 0 \\ & 0 & & -\sin\alpha & \cos\alpha & 0 \\ & & & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{K}^e = T^{eT} \cdot K^e \cdot T^e \qquad \bar{P} = T^{eT} P^e$$

$$\bar{K}^e \bar{q}^e = \bar{P}^e$$

(bending + tensil + torsion)

DOF = 12. $\quad q^e = [u_1 \; v_1 \; w_1 \; \theta_{x1} \; \theta_{y1} \; \theta_{z1} \; u_2 \; v_2 \; w_2 \; \theta_{x2} \; \theta_{y2} \; \theta_{z2}]^T$

$$K^e_{u_1,u_2} = \frac{EA}{l} \begin{array}{cc} u_1 & u_2 \\ \left[\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}\right] & \begin{matrix} u_1 \\ u_2 \end{matrix} \end{array}$$

$$K^e_{\theta_1 \theta_2} = \frac{GJ}{l} \begin{array}{cc} \theta_{x1} & \theta_{x2} \\ \left[\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}\right] & \begin{matrix} \theta_{x1} \\ \theta_{x2} \end{matrix} \end{array}$$

G – shear modulus

J – torsion moment of inertia

$$K^e(O_{xy}) = \frac{EI_z}{l^3} \begin{array}{c} \begin{matrix} v_1 & \theta_{z1} & v_2 & \theta_{z2} \end{matrix} \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \end{array}$$

similarly

$$K^e(O_{xz}) = \frac{EI_y}{l^3} \begin{array}{cc} \begin{matrix} w_1 & \theta_{y1} & w_2 & \theta_{y2} \end{matrix} & \\ \begin{bmatrix} 12 & -6l & -12 & -6l \\ -6l & 4l^2 & +6l & 2l^2 \\ -12 & +6l & 12 & +6l \\ -6l & 2l^2 & +6l & 4l^2 \end{bmatrix} & \begin{matrix} w_1 \\ \theta_{y1} \\ w_2 \\ \theta_{y2} \end{matrix} \end{array}$$

$$K^e = \begin{array}{c} \begin{matrix} u_1 & v_1 & w_1 & \theta_{x1} & \theta_{y1} & \theta_{z1} & u_2 & v_2 & w_2 & \theta_{x2} & \theta_{y2} & \theta_{z2} \end{matrix} \\ \begin{bmatrix} \\ & ? & \\ \\ \end{bmatrix} \end{array}$$

$$T^e = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \text{ where } \lambda = \begin{bmatrix} \cos(x,\bar{x}) & \cos(x,\bar{y}) & \cos(x,\bar{z}) \\ \cos(y,\bar{x}) & \cos(y,\bar{y}) & \cos(y,\bar{z}) \\ \cos(z,\bar{x}) & \cos(z,\bar{y}) & \cos(z,\bar{z}) \end{bmatrix}$$

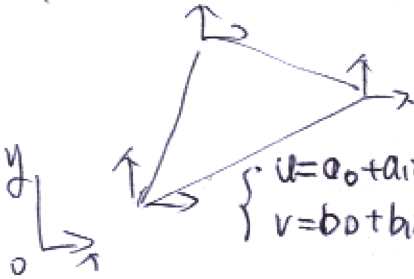
$\lambda_{ij} = u_i \cdot \bar{u}_j$

unit vector.

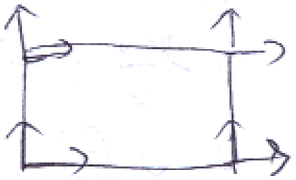
continuum structure, approximated discretization →

beam & Truss, natural discretization

Plane Problems



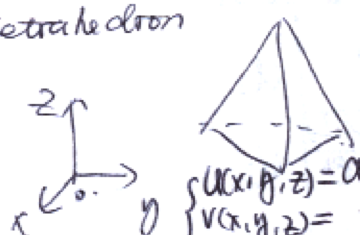
$$\begin{cases} u = a_0 + a_1 x + a_2 y \\ v = b_0 + b_1 x + b_2 y \end{cases}$$



$$\begin{cases} u(x,y) = a_0 + a_1 x + b_2 y + a_3 xy \\ v(x,y) = b_0 + b_1 x + b_2 y + b_3 xy \end{cases}$$

3D Space problems.

Tetrahedron



$$\begin{cases} u(x,y,z) = a_0 + a_1 x + a_2 y + a_3 z \\ v(x,y,z) = \cdots \\ w(x,y,z) = \cdots \end{cases}$$

Hexahedron

$$\begin{cases} u(x,y,z) = \cdots \\ v(\quad) = \cdots \\ w(\quad) = \cdots \end{cases}$$

② ~~xxxxxx~~ Potential Energy

$\therefore \Pi = \frac{1}{2} q^T K q - P^T q = -\frac{1}{2} q^T K q = -U$

strain energy.

$\Pi_{appr} > \Pi_{exact}$

$\therefore U_{appr} < U_{exact}$ $\quad K_{appr} q_{appr} = P = K_{exact} q_{exact}$.

$q^T_{appr} P < q^T_{exact} P$.

switch order

① . h-method, more node. convergence ✗ stable ✓

p-method, higher order basis ✓ ✗.

$\vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3$

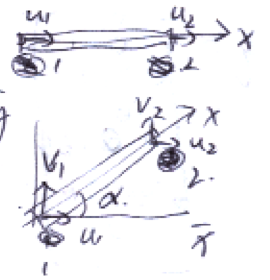
$u(x) = a_1 + a_2 x + a_3 x^2$.

r-method, move nodes.

adaptive method.

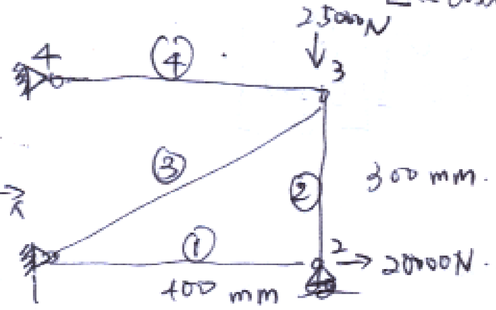
$$
K^e = \begin{array}{c}
\begin{array}{cccccccccccc} u_1 & v_1 & w_1 & \theta_{x1} & \theta_{y1} & \theta_{z1} & u_2 & v_2 & w_2 & \theta_{x2} & \theta_{y2} & \theta_{z2} \end{array} \\
\begin{array}{cc}
\begin{bmatrix}
\frac{EA}{\ell} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{\ell} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{\ell^3} & 0 & 0 & 0 & \frac{6EI_z}{\ell^2} & 0 & -\frac{12EI_z}{\ell^3} & 0 & 0 & 0 & \frac{6EI_z}{\ell^2} \\
0 & 0 & \frac{12EI_y}{\ell^3} & 0 & -\frac{6EI_y}{\ell^2} & 0 & 0 & 0 & -\frac{12EI_y}{\ell^3} & 0 & -\frac{6EI_y}{\ell^2} & 0 \\
0 & 0 & 0 & \frac{GJ}{\ell} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{\ell} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{\ell^2} & 0 & \frac{4EI_y}{\ell} & 0 & 0 & 0 & \frac{6EI_y}{\ell^2} & 0 & \frac{2EI_y}{\ell} & 0 \\
0 & \frac{6EI_z}{\ell^2} & 0 & 0 & 0 & \frac{4EI_z}{\ell} & 0 & -\frac{6EI_z}{\ell^2} & 0 & 0 & 0 & \frac{2EI_z}{\ell} \\
-\frac{EA}{\ell} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{\ell} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_z}{\ell^3} & 0 & 0 & 0 & -\frac{6EI_z}{\ell^2} & 0 & \frac{12EI_z}{\ell^3} & 0 & 0 & 0 & -\frac{6EI_z}{\ell^2} \\
0 & 0 & -\frac{12EI_y}{\ell^3} & 0 & \frac{6EI_y}{\ell^2} & 0 & 0 & 0 & \frac{12EI_y}{\ell^3} & 0 & \frac{6EI_y}{\ell^2} & 0 \\
0 & 0 & 0 & -\frac{GJ}{\ell} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{\ell} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{\ell^2} & 0 & \frac{2EI_y}{\ell} & 0 & 0 & 0 & \frac{6EI_y}{\ell^2} & 0 & \frac{4EI_y}{\ell} & 0 \\
0 & \frac{6EI_z}{\ell^2} & 0 & 0 & 0 & \frac{2EI_z}{\ell} & 0 & -\frac{6EI_z}{\ell^2} & 0 & 0 & 0 & \frac{4EI_z}{\ell}
\end{bmatrix}
&
\begin{array}{c} u_1 \\ v_1 \\ w_1 \\ \theta_{x1} \\ \theta_{y1} \\ \theta_{z1} \\ u_2 \\ v_2 \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \\ \theta_{z2} \end{array}
\end{array}
\end{array}
$$

Truss $\mathbb{K}^e = \frac{EA}{e}\begin{array}{c c} u_1 & u_2 \\ \end{array}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$



9

$$\bar{\mathbb{K}}^e = \frac{EA}{e}\begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array}$$
$$\begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha & -\cos^2\alpha & -\cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha & -\cos\alpha\sin\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -\cos\alpha\sin\alpha & \cos^2\alpha & \cos\alpha\sin\alpha \\ -\cos\alpha\sin\alpha & -\sin^2\alpha & \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix}$$



$E = 29.5 \times 10^4$ N/mm

$A = 100$ mm^2.

$$\bar{\mathbb{K}}^{(1)} = \frac{EA}{400}\begin{array}{c|c} & u_1 \quad v_1 \quad u_2 \quad v_2 \\ \hline \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix} & \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = \frac{EA}{6000}\begin{bmatrix} 15 & 0 & -15 & 0 \\ 0 & 0 & 0 & 0 \\ -15 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\mathbb{K}}^{(2)} = \frac{EA}{300}\begin{array}{c|c} & u_2 \quad v_2 \quad u_3 \quad v_3 \\ \hline \begin{matrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} = \frac{EA}{6000}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 20 & 0 & -20 \\ 0 & 0 & 0 & 0 \\ 0 & -20 & 0 & 20 \end{bmatrix}$$

$(\alpha = 90°)$.

$$\bar{\mathbb{K}}^{(3)} = \frac{EA}{500}\begin{array}{c|c} & u_1 \quad v_1 \quad u_3 \quad v_3 \\ \hline \begin{matrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{matrix} & \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix} \end{array} = \frac{EA}{6000}\begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$\left(\begin{matrix}\sin\alpha = 0.6 \\ \cos\alpha = 0.8\end{matrix}\right)$

$$\bar{\mathbb{K}}^{(4)} = \frac{EA}{400}\begin{array}{c|c} & u_4 \quad v_4 \quad u_3 \quad v_3 \\ \hline \begin{matrix} u_4 \\ v_4 \\ u_3 \\ v_3 \end{matrix} & \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = \frac{EA}{6000}\begin{bmatrix} 15 & 0 & -15 & 0 \\ 0 & 0 & 0 & 0 \\ -15 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{EA}{6000}\begin{array}{c} u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4 \\ \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} \end{array}\begin{bmatrix} \cancel{u_1}^{\,0} \\ \cancel{v_1}^{\,0} \\ u_2 \\ \cancel{v_2}^{\,0} \\ u_3 \\ v_3 \\ \cancel{u_4}^{\,0} \\ \cancel{v_4}^{\,0} \end{bmatrix} = \begin{bmatrix} R_{x1} \\ R_{y1} \\ 20000 \\ R_{y2} \\ 0 \\ -25000 \\ R_{x4} \\ R_{y4} \end{bmatrix}$$

solve for displacement: $20 + 0.36 \times 12 + 0$

0.48×12

symmetric

$$\begin{array}{c} u_2 \quad u_3 \quad v_3 \\ \begin{bmatrix} 15 & 0 & 0 \\ 0 & 0.64\times12+15 & \\ 0 & 0.48\times12 & \end{bmatrix} \end{array}\begin{bmatrix} u_2 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 20,000 \\ \cancel{25,000}\ 0 \\ -25,000 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 0 & 0 \\ 0 & 22.68 & 5.76 \\ 0 & 5.76 & 24.32 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 20,000 \\ 0 \\ -25,000 \end{bmatrix} \Rightarrow \begin{bmatrix} u_2 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0.2712 \\ 0.0565 \\ -0.2225 \end{bmatrix} \text{ mm.}$$

displacement $q = \begin{bmatrix} 0 \\ 0 \\ 0.2712 \\ 0.0565 \\ -0.2225 \\ 0 \\ 0 \end{bmatrix}$ mm.

~~stress~~ strain $\varepsilon^{(1)} = B^{(1)} T^{(1)} q^{(1)} = \begin{bmatrix} -\frac{1}{400} & \frac{1}{400} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.2712 \\ 0 \end{bmatrix}$

$= 0.000678$

stress $\sigma^{(1)} = E B^{(1)} T^{(1)} q^{(1)} = E \varepsilon^{(1)} = 200$ N/mm²

similarly, $\sigma^{(2)} = -218.8$ N/mm²

$\sigma^{(3)} = 52.08$ N/mm²

$\sigma^{(4)} = 41.67$ N/mm²

RF

$$\begin{bmatrix} R_{x1} \\ R_{y1} \\ R_{y2} \\ R_{x4} \\ R_{y4} \end{bmatrix} = \frac{EA}{6000} \begin{array}{c} \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 15+12\times0.64 & 0.48\times12 & -15 & 0 & -7.68 & -5.76 & 0 & 0 \\ 0.48\times12 & 0.36\times12 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & -20 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ v_2 \\ u_4 \\ v_4 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0.2712 \\ 0 \\ 0.565 \\ -0.2225 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -15.833 \\ 3.126 \\ 21.879 \\ -41.67 \\ 0 \end{bmatrix} \text{ N}$$